The University of Nottingham

DEPARTMENT OF Mechanical Engineering

A LEVEL 2 MODULE, Spring SEMESTER 2020-2021

MMME2046 Dynamics and Controls

Time allowed 3 hours plus 30 minutes upload period

Open-book take-home examination

Answer ALL questions

You must submit a single pdf document, produced in accordance with the guidelines provided on take-home examinations, that contains all of the work that you wish to have marked for this open-book examination. Your submission file should be named in the format '[Student ID]_[Module Code].pdf'.

Write your student ID number at the top of each page of your answers.

This work must be carried out and submitted as described on the Moodle page for this module. All work must be submitted via Moodle by the submission deadline. **Work submitted after the deadline will not be accepted without a valid EC.**

No academic enquiries will be answered by staff and no amendments to papers will be issued during the examination. If you believe there is a misprint, note it in your submission but answer the question as written.
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Answer all questions on this exam

1. A crankshaft-piston mechanism is shown in Fig.Qx and is driven by an electric motor inputting a torque T at point A. At the instant shown link AB is rotating clockwise at a constant angular velocity of ω_{AB}=33rad/sec and it is known that $\gamma=15^o$ and $\delta=45^o$. It is also given that AB=40mm and BC=150mm. The piston's mass is equal to 0.3kg and it can be considered as a particle. Gravitational acceleration is g=9.81m/sec^2. Link AB has mass m_{AB}=0.1kg while link BC is of negligible mass. The friction coefficient between the piston and the chamber walls is equal to μ=0.1.

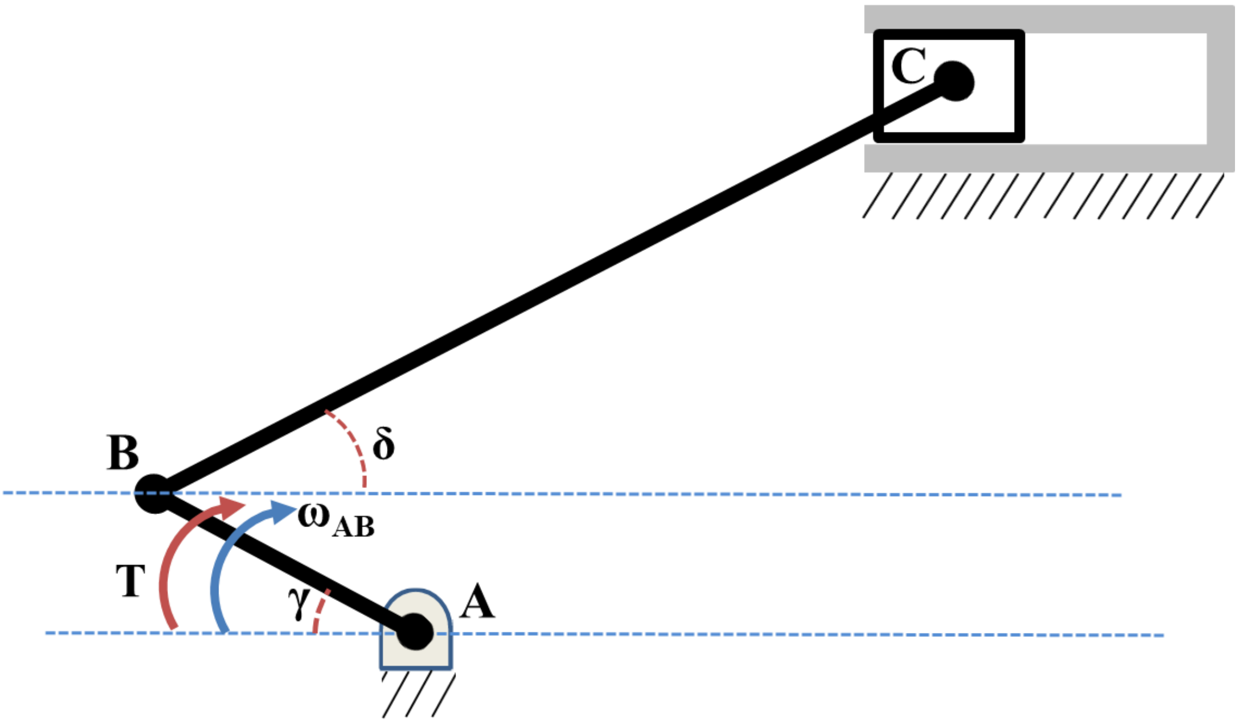
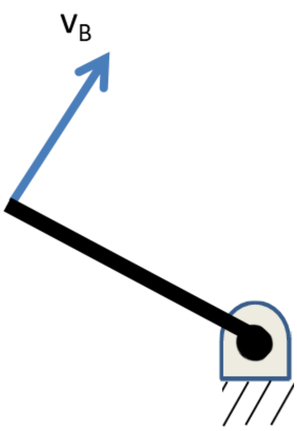


FIGURE Q1

a) This is a common slider-crank mechanism, having a single degree of freedom. +2

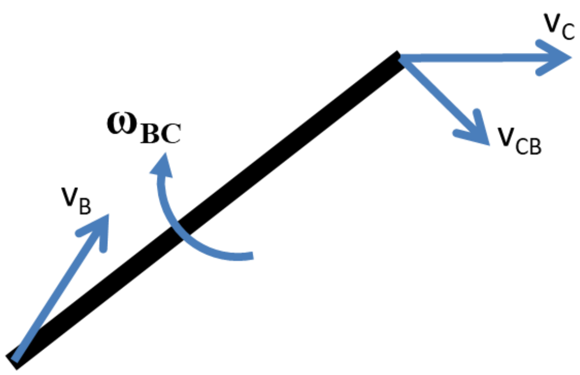
b) The velocity at point B is perpendicular to AB and has a magnitude:

$v_B = \omega_{AB} \cdot AB = 1.32$ m/s +1



+1

c) Resolving perpendicularly to the 2nd unknown which is the velocity of point C should give us enough information to solve for ω_{BC}. According to the figure below we have:



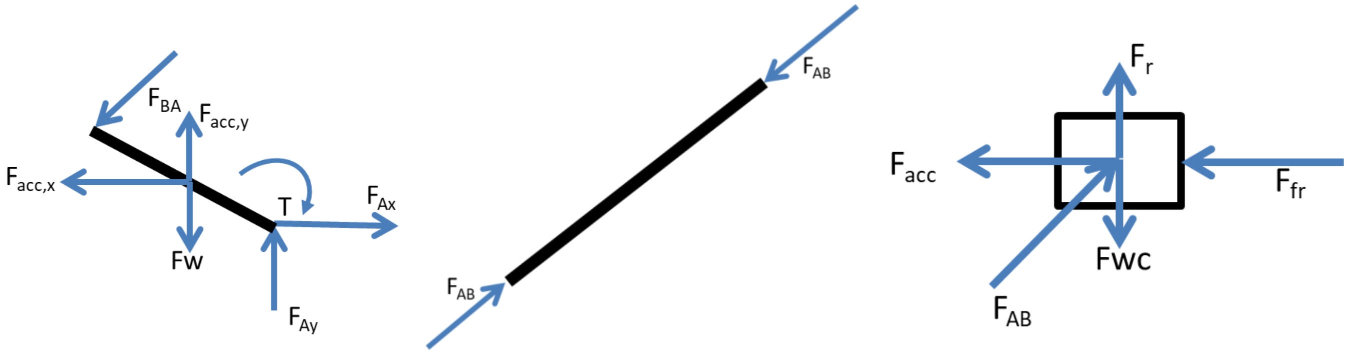
+1

$\underline{v}_C=\underline{v}_B+\underline{v}_{CB}$ +1

Therefore: $v_{Cy}=v_{By}+v_{CBy}=v_B\cos(\gamma)+v_{CB}\cos(90-\delta+90)= v_B\cos(\gamma)+ \omega_{BC}BC\cos(90-\delta+90)$ therefore

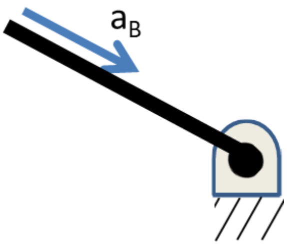
$\omega_{BC}=- v_B\cos(\gamma) / BC\cos(90-\delta+90) = 12.02$ rad/s +3

d)



+1 for BC, +2 for AB, +3 for piston

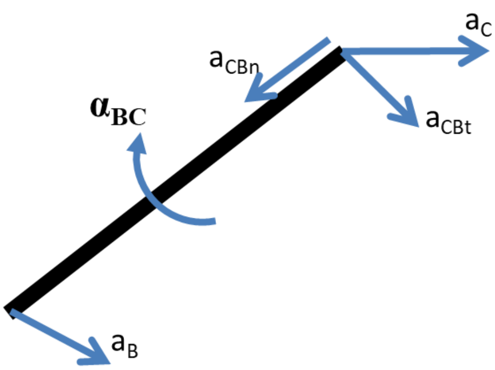
e) Point B has only a normal acceleration component: $a_B= \omega_{AB}^2{*}AB=43.6$ m/sec^2 as shown below. +4



$a_{Bx}=a_B\cos(90-\gamma)=11.27$ m/sec^2

$a_{By}=a_B\cos(90-\gamma-90)=42.07$ m/sec^2

f) According to Chasles' theorem: $\underline{a}_C=\underline{a}_B+\underline{a}_{CBn}+\underline{a}_{CBt}$ +1



We choose to solve the problem by resolving the above acceleration equation perpendicularly to the 2nd unknown which is $\underline{a}_{CBt}$

$a_{CBn}=\omega_{BC}^2{*}BC=21.68$ m/sec^2 +1

Resolving the Chasles' equation towards a_{CBn} gives

$\underline{a}_C\cos(\delta+180)=\underline{a}_B\cos(\gamma+\delta+180)+\underline{a}_{CBn}+0$ +3

Therefore $\underline{a}_C=(\underline{a}_B\cos(\gamma+\delta+180)+\underline{a}_{CBn}) / \cos(\delta+180) = -0.73m/s^2$ accelerating towards the left hand side.

g) We can easily compute from the piston's FBD: $Fwc=m_c g=2.94N$, $Facc=m_c*a_c= -0.22N$ and $Ffr=\mu*(Fwc-F_{AB}\cos(90-\delta))$ +2 Therefore:

ΣFx: $F_{AB}\cos(\delta)-Facc-Ffr=0$ therefore $F_{AB}\cos(\delta)-Facc- \mu*(Fwc-F_{AB}\cos(90-\delta)) =$

$=F_{AB}\cos(\delta)-Facc- \mu*Fwc - \mu*F_{AB}\cos(90-\delta)$ +2

$F_{AB}(\cos(\delta) - \mu*\cos(90-\delta)) -Facc- \mu*Fwc =0$ and

$F_{AB}= [Facc + \mu*Fwc] / (\cos(\delta) - \mu*\cos(90-\delta)) = 0.12N.$ +2

2. The system in figure 1is subjected to a unit step input at t=0. If $G_p(s) = \frac{1}{4s+10}$ determine:

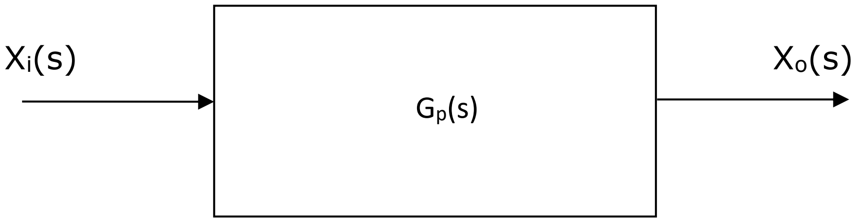


FIGURE Q2.1

a. The steady state error as $t \to \infty$.

(3 marks)

Solution: For an input $X_i(s) = \frac{1}{s}$, $X_o(s) = \frac{1}{s}\left(\frac{1}{4s+10}\right)$
Partial fractions give $X_o(s) = \frac{1}{s}\left(\frac{1}{4s+10}\right) = \frac{0.1}{s} - \frac{0.4}{4s+10}$
Inverse Laplace transforms then give

$$x_o(t) = 0.1(1 - e^{-2.5t})$$

And the steady state error is given by:

$$\lim_{t\to\infty}\left(x_i(t) - x_o(t)\right) = 1 - 0.1(1 - e^{-2.5t}) = 0.9$$

The same answer can be obtained using the final value theorem:

$$\lim_{s\to 0} s\left(X_i(s) - X_o(s)\right) = s\left(\frac{1}{s} - \frac{1}{s}\left(\frac{1}{4s+10}\right)\right) = 0.9$$

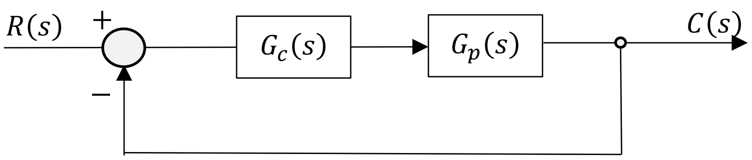
b. The time taken for the system to be at 95% of the final (steady state) level.

(7 marks)

Solution: From the time domain solution in part (a):

$$x_o(t) = 0.1(1 - e^{-2.5t})$$
$$1 - e^{-2.5t} = 0.95 \text{ when } e^{-2.5t} = 0.05$$
$$-2.5t = \ln 0.05 = -3.00$$
$$t = 1.20$$

This system is integrated with a feedback controller as shown in figure 2.



c. What is the transfer function $\frac{C(s)}{R(s)}$ of the system in terms of $G_c(s)$ and $G_p(s)$?

(4 marks)

Solution: $C(s) = \left(R(s) - C(s)\right)\left(G_c(s)G_p(s)\right)$

$$\frac{C(s)}{R(s)} = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)}$$

Note to moderator: Students may derive or quote this result as it will be familiar to most from the example sheets and lectures.

d. For $G_c(s) = 3s + k$, find the value of k that reduces the steady state error when the system is subject to a unit step input to 5%.

(4 marks)

Solution: Overall transfer function for the system:

$$\frac{C(s)}{R(s)}=\frac{\frac{3s+k}{4s+10}}{1+\frac{3s+k}{4s+10}}=\frac{3s+k}{3s+k+4s+10}=\frac{3s+k}{7s+(10+k)}$$

Unit step response (i.e. $R(s)=\frac{1}{s}$) can be found using the final value theorem:

$$\lim_{s\to 0} s\big(C(s)\big)=s\left(\frac{1}{s}\left(\frac{3s+k}{7s+(10+k)}\right)\right)=\frac{k}{10+k}$$

The error as t tends to infinity is given by:

$$e(t)=1-\lim_{s\to 0} s\big(C(s)\big)=1-\frac{k}{10+k}=\frac{10}{10+k}$$

And for the error to be 5%, the value of k must be 190.

$$\frac{10}{10+k}=0.05=\frac{1}{20}$$
$$200=10+k$$
$$k=190$$

To improve the system response further, a PID controller is implemented for which:

$$G_c(s)=\frac{3s^2+ks+2}{s}$$

e. What is the effect on the steady state error for a unit step input for k= 10? The plant transfer function $G_p(s)=\frac{1}{4s+10}$.

(4 marks)

Solution:

$$\frac{C(s)}{R(s)}=\frac{G_c(s)G_p(s)}{1+G_c(s)G_p(s)}=\frac{\left(\frac{3s^2+ks+2}{s}\right)\left(\frac{1}{4s+10}\right)}{1+\left(\frac{3s^2+ks+2}{s}\right)\left(\frac{1}{4s+10}\right)}$$

$$\frac{C(s)}{R(s)}=\frac{3s^2+ks+2}{4s^2+10s+3s^2+ks+2}=\frac{3s^2+ks+2}{7s^2+(k+10)s+2}$$

K=10 so

$$\frac{C(s)}{R(s)}=\frac{3s^2+10s+2}{7s^2+20s+2}$$

The steady state error is therefore

$$e(t)=1-\lim_{s\to 0} s\big(C(s)\big)=1-\lim_{s\to 0} s\left(\frac{1}{s}\left(\frac{3s^2+10s+2}{7s^2+20s+2}\right)\right)=0$$

The effect of the PID controller is therefore to eliminate the steady state error.

f. What is the range of values of k for which the system will be stable?

(4 marks)

Solution:

Note to moderator: I would expect many of the weaker students to go straight from the denominator $(7s^2+(k+10)s+2)$of the transfer function to the Routh-Hurwitz criterion and then to show that the system is stable when k>-10 even though this is not strictly necessary – the system is stable for all k>-10 recognising that the denominator of the transfer function corresponds to the Laplace transform pair:

$$\frac{\omega}{\sqrt{1-\gamma^2}}e^{-\gamma\omega t}\sin\left(\omega t\sqrt{1-\gamma^2}\right)\xrightarrow{\mathcal{L}}\frac{\omega^2}{s^2+2\gamma\omega s+\omega^2}$$

For any $\gamma>0$ the system will be stable, hence k>-10. (5 marks)

g. Figure Q2.3 shows the root locus for a system with the overall transfer function

$$G(s)=\frac{\omega^2}{s^2+2\gamma\omega s+\omega^2}$$

Describe the system behaviours you would expect to see in response to a step input for the roots a, b, c and d in the graph.

[4]

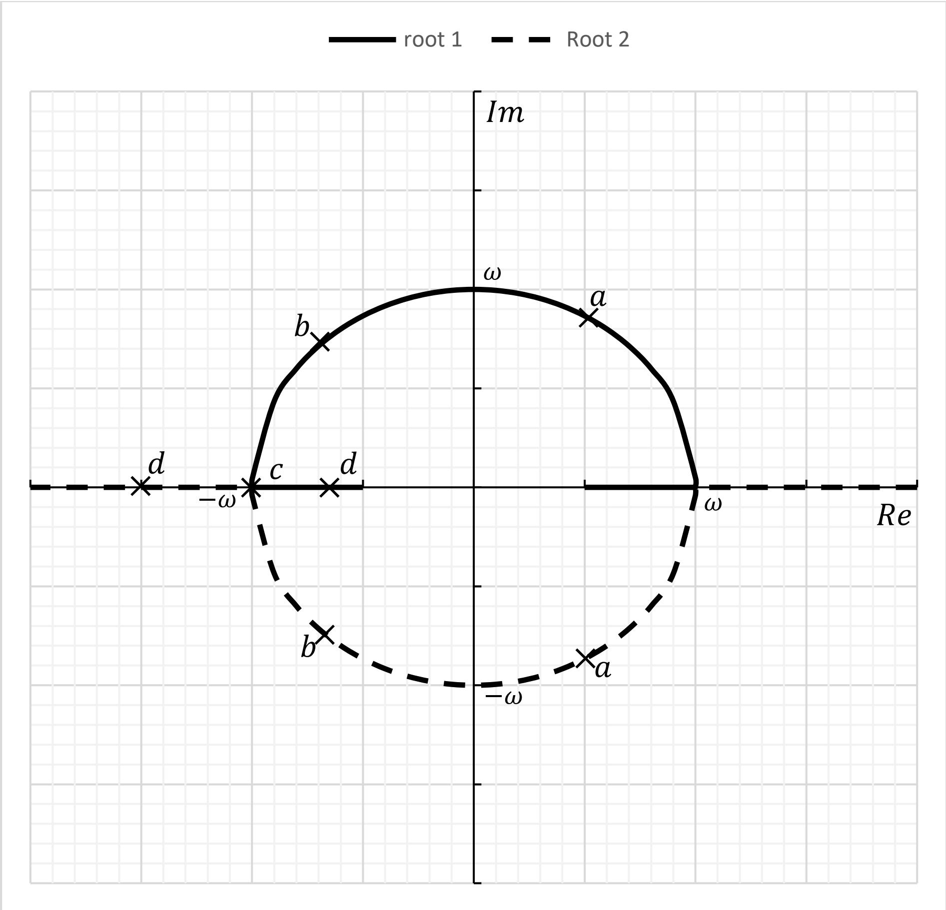


Figure Q2.3

Solution:

The pair of roots at (a) are unstable – the system will oscillate and the oscillations will increase in magnitude with time.

The pair of roots at (b) are underdamped – the system will oscillate and the oscillations will decay with time and the system will reach a stable steady state value.

The root at (c) is critically damped. The system will not oscillate or overshoot and will reach a stable value rapidly.

The roots at (d) are overdamped. The system will respond more slowly to the input and will take longer to reach 95% of the steady state value.

Mark scheme: ½ mark for the correct term (unstable, underdamped, critically damped, overdamped) and ½ mark for the description, key terms are underlined although paraphrases will be accepted.

1. Figure Q3.1 shows a rigid beam AD with moment of inertia I_o about the pivot at point B. The beam is supported at D by a vertical spring having stiffness *k* and at A by a damper having damping coefficient *c*. θ denotes the angular displacement of the beam about the horizontal equilibrium state and is assumed positive in the clockwise direction as shown. All displacements and rotations in this system are assumed to be small.

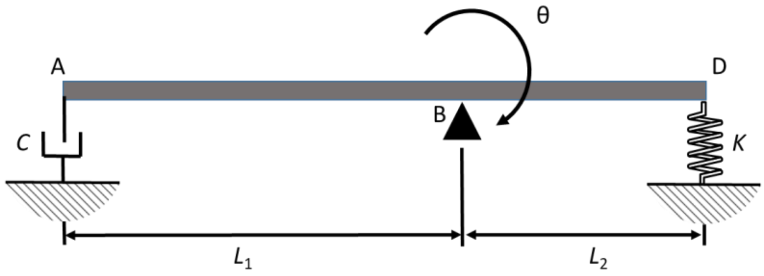
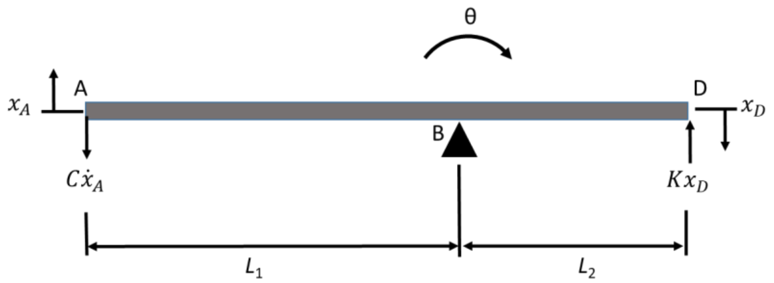


FIGURE Q3.1

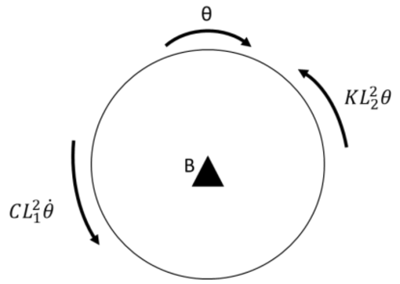
$I_o = 0.5$ kg m^2
$k = 1000$ N/m
$c = 10$ Ns/m
$L_1 = 1$ m
$L_2 = 0.5$ m

a) Draw the Free Body Diagram for this system. This or ones with the circle are both acceptable. [3]

Don't forget the spring and damper are linear, so in the FBD below they show as linear forces, but when transferring to EOM in part b) they'll need to become torques.



or



b) Derive the equation of motion for rotation of the beam, *θ(t)*, about the pivot A. [3]

Relationships for linear displacements at springs to rotational displacements (students may have included this in part A).

$$x_A = L_1\theta$$
$$\dot{x}_A = L_1\dot{\theta}$$
$$x_D = L_2\theta$$

Keeping in mind the forces shown in the diagram for the spring and damper are linear, so they will have to be converted to torques the EOM now becomes…

$$I_o\ddot{\theta} = -CL_1^2\dot{\theta} - KL_2^2\theta$$

$$I_o\ddot{\theta} + CL_1^2\dot{\theta} + KL_2^2\theta = 0$$

$$0.5\ddot{\theta} + 10\dot{\theta} + 250\theta = 0$$

[1pt for newton's; 1 pt for in correct form; 1 pt for correct derivation]

c) What are the undamped natural frequency ω_n and damping ratio γ for this system? [4]

[2 pts each]

$$\omega_n = \sqrt{\frac{KL_2^2}{I_o}} = 22.4\frac{rad}{s} = 3.6Hz$$

$$\gamma = \frac{CL_1^2}{2\sqrt{KL_2^2 I_o}} = 0.45$$

d) If the beam is lifted vertically up by 0.1m at point A and then released from rest determine the resulting transient angular displacement at B as a function of time, $\theta_{tr}(t)$. What frequency will this system vibrate at after being released from rest? [8]

The damping ratio is less than 1, so this is an underdamped system and the following equations apply from the formula sheet.

$$z(t)_{tr} = e^{-\gamma\omega_n t}[B_1 \cos(\Omega_n t) + B_2 \sin(\Omega_n t)]$$

$$\dot{z}(t)_{tr} = B_1 e^{-\gamma\omega_n t}[-\Omega_n \sin(\Omega_n t) - \gamma\omega_n \cos(\Omega_n t)] + B_2 e^{-\gamma\omega_n t}[\Omega_n \cos(\Omega_n t) - \gamma\omega_n \sin(\Omega_n t)]$$

Where the vibration frequency of the system will be:

$$\Omega_n = \omega_d = \omega_n\sqrt{1-\gamma^2} = 20\frac{rad}{s} = 3.2\ Hz$$

[2; 1 pt if not in Hz]

For initial conditions where x_o=0.1m at point A the rotation will be $\theta_o = \frac{0.1}{L_1} = 0.1\ rad$ and $t = 0$.

$$\theta(t)_{tr} = 0.1 = 1[B_1 * 1 + B_2 * 0)]$$
$$B_1 = 0.1$$

[2]

For initial conditions where $\dot{\theta}(0)_{tr} = 0$ and $t = 0$.

$$\dot{\theta}(t)_{tr} = 0 = 0.1[-\gamma\omega_n] + B_2[\Omega_n]$$
$$B_2 = \frac{0.1\gamma\omega_n}{\omega_n\sqrt{1-\gamma^2}} = 0.05$$

[3]

Therefore

$$\theta(t)_{tr} = e^{-10t}[0.1 \cos(20t) + 0.05 \sin(20t)]$$

[1]

A harmonic moment, $M(t) = Me^{i\omega t} = 10e^{18.9it}$, is now applied at point B as shown in Figure Q3.2 and the system allowed to come to steady state motion.

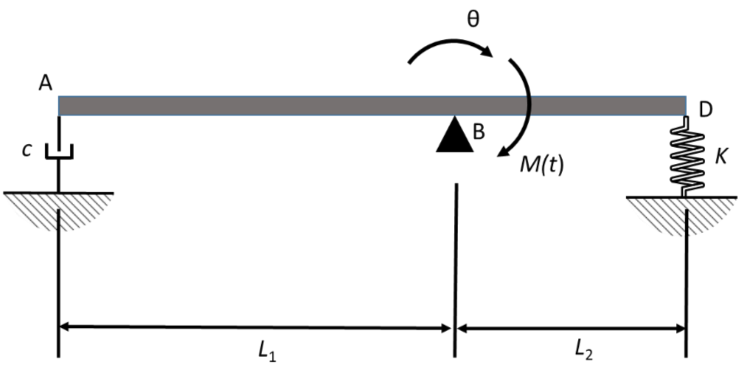
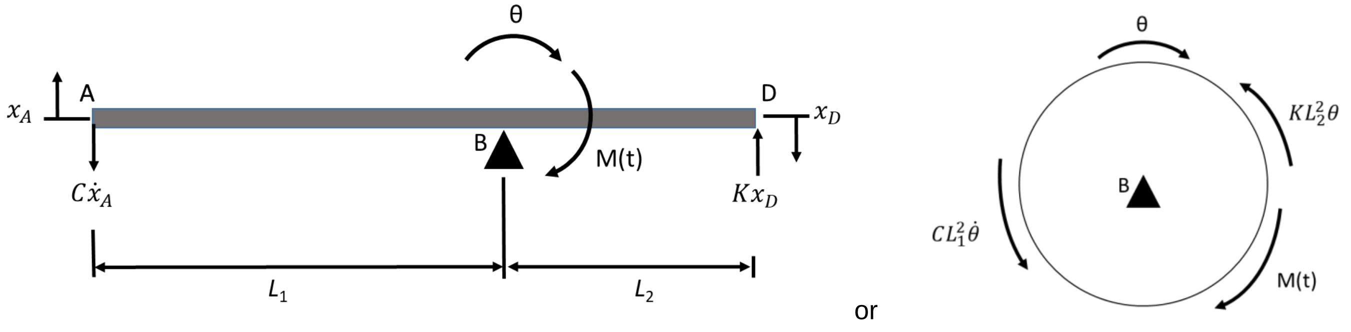


FIGURE Q3.2

e) Derive the frequency response function, $\frac{\Theta^*}{M}$, for the displacement of the beam under the excitation force $M(t) = Me^{i\omega t}$. [4]

You can find the solution from the formula sheet or derive as follows:

Start with the EOM for the new system (students might find it useful to update the FBD with the new Moment, M(t), but really it's the EOM we're concerned with so no points are assigned to the FBD).



$$I_o\ddot{\theta} + CL_1^2\dot{\theta} + KL_2^2\theta = M(t)$$

[1]

Keep in mind the substitutions

$$M(t) = Me^{i\omega t}$$
$$\theta(t) = \Theta^* e^{i\omega t}$$
$$\dot{\theta}(t) = \mathrm{i}\omega\Theta^* e^{i\omega t}$$
$$\ddot{\theta}(t) = -\omega^2\Theta^* e^{i\omega t}$$

$$-I_o\omega^2\Theta^* e^{i\omega t} + CL_1^2\mathrm{i}\omega\Theta^* e^{i\omega t} + KL_2^2\Theta^* e^{i\omega t} = Me^{i\omega t}$$

[1]

Exponents will drop out, then rearrange for the FRF

$$(-I_o\omega^2 + CL_1^2\mathrm{i}\omega + KL_2^2)\Theta^* = M$$

$$H(\omega) = \frac{\Theta^*}{M} = \frac{1}{(KL_2^2 - I_o\omega^2 + CL_1^2\mathrm{i}\omega)} = \frac{1}{(72.3 + 188.5\mathrm{i})}$$

[2]

f) Calculate the magnitude of the steady state response, $|\theta^*|$, and its phase relative to the applied moment, α. E.g. determine $\theta_{ss}(t) = |\theta^*|cos(18.9t + \alpha)$. [4]

Taking the FRF from the previous equation and solving for the magnitude of a complex number will result in:

$$|\Theta^*| = \sqrt{\frac{(M)^2}{(KL_2^2 - I_o\omega^2)^2 + (CL_1^2\omega)^2}} = \sqrt{\frac{(10)^2}{(250 - 177.6)^2 + (188.5)^2}}$$
$$= \sqrt{\frac{(10)^2}{(72.3)^2 + (188.5)^2}} = 0.05\, rad = 2.8°$$

[2; rad or degrees accepted]

$$\alpha = tan^{-1}\left(\frac{-\omega CL_1^2}{KL_2^2 - I_o\omega^2}\right) = tan^{-1}\left(\frac{188.5}{250 - 177.6}\right) = -1.2\, rad = -69°$$

[2; rad or degrees accepted]

You can get the same thing if you re-derive from scratch or use the formula sheet. Any of these are acceptable ways to get the same answer.

g) Determine the amplitude of force acting on the ground through the spring only. [8]

To do this you need a second equation for the force acting through the spring on the ground, q(t).

$$q(t) = KL_2\theta$$

[1]

Substitute in the following expressions to the EOM and $q(t)$.

$$M(t) = Me^{i\omega t}$$
$$q(t) = Q^*e^{i\omega t}$$
$$\theta(t) = \Theta^*e^{i\omega t}$$
$$\dot{\theta}(t) = \mathrm{i}\omega\Theta^*e^{i\omega t}$$
$$\ddot{\theta}(t) = -\omega^2\Theta^*e^{i\omega t}$$

[2]

The exponential terms will nicely drop out, and you can rearrange the two equations to

$$\Theta^* = \frac{M}{(KL_2^2 - I_o\omega^2) - iCL_1^2\omega}$$

$$Q^* = KL_2\Theta^*$$

Combining the two equations to eliminate Θ* results in a relationship for the transmitted force:

$$T_F = \frac{Q^*}{M} = \frac{KL_2}{(KL_2^2 - I_o\omega^2) + iCL_1^2\omega}$$

[2]

Find the magnitude of Q*

$$|Q^*| = M\sqrt{\frac{(KL_2)^2}{(KL_2^2 - I_o\omega^2)^2 + (CL_1^2\omega)^2}} = 10\sqrt{\frac{(250)^2}{(250 - 177.6)^2 + (188.5)^2}} = 24.7N$$

[2]

h) If the system is experiencing unwanted levels of motion when operating at a given frequency, what is one way you might reduce the amplitude of motion at this frequency? What is one drawback of the method you have chosen? [6]

There are going to be many potential answers here, but the ones I would expect are:

i. Change damping, stiffness and/or mass (this will depend where you are in relation to the natural frequency and since I don't give that in the question anything goes really)
ii. Lower the natural frequency to push the operating point further into the attenuation region. This could be accomplished by decreasing stiffness or increasing the mass of the beam.
iii. Add some sort of controller and means to implement it.
iv. Operate at another frequency where the motion will be attenuated.

Drawbacks will depend on their answer above, and could be rather varied. So things to look out for are:

- Acknowledging that changes to M, C and/or K could change the overall system and thus require further evaluation of the structure. For example, adding mass might result in the structure no longer supporting itself, or a softer spring might mean you exceed the springs static displacement kind of thing, or in the attenuation region higher damping results in greater displacements, etc.
- Changes at one frequency might also affect performance at other frequencies.
- Adding a controller can have a few drawbacks, but probably the most significant one is simply the increase in complexity and need for additional actuation elements (adding actuation then goes back to the structural integrity point as well).
- You can't always operate at another frequency.
- Cost and/or complexity could increase.

Many other options are possible as the question is purposefully opened ended.